The Effects of Risk Aversion and Money Illusion on the Endogenous Dividend Growth Rate.

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Abstract

We evaluate the impact of risk aversion and money illusion in the equity and options markets when the expected dividend growth rate is endogenously determined as a function of the dividend-price ratio and expected inflation. The closed-form equilibrium expressions for the dividend-price ratio, expected inflation, and dividend growth rate allow us to perform comparative statics to understand their sensitivity relative to the agent’s preference parameters. In our calibration exercise, the dividend-price ratio is an increasing function of risk aversion and money illusion, expected inflation is much less sensitive to variations on these parameters, and zero-coupon caplet prices are affected in opposite directions by them.

Keywords: Dividend Growth, Asset Pricing, Money Illusion, CPI Derivatives.

\textit{JEL:} G12, G13, G30

1. Introduction

For the past three decades, financial economists have relentlessly investigated the components of dividend growth rate. While the seminal paper of Campbell and Shiller (1988) establishes the general consensus that the dividend-price ratio is not a relevant component of dividend growth, a growing number of studies have been questioning this idea and presenting convincing evidence that some variables are important factors to understand dividend growth. For instance, Chen (2009) challenges the idea that the dividend-price ratio does not predict dividend growth and presents evidence of predictability for the era
before the Second World War under the assumption that the monthly dividends are not
reinvested in financial markets; a result also verified by Chen et al. (2012) and Zhu et al.
(2018). Another important factor is presented by Engsted and Pedersen (2010). The
authors show that the long-horizon dividend growth predictability depends on whether
the returns and dividend yields are measured in nominal or real terms. Thus, accounting
for expected inflation can substantially increase the predictability of dividend growth.

While these two factors — dividend-price ratio and expected inflation — have been
proved to be important components of dividend growth, most of the existing general equi-
librium models in the literature cannot generate a mechanism where these two components
impact dividend growth. Since the starting point of exchange economies \textit{à la} Lucas (1978)
is to postulate an exogenous process for the dividend process, this framework precludes
any analysis on how changes in agents’ characteristics, such as risk aversion and money il-
lusion, impact the equilibrium expected inflation and dividend-price ratio and, ultimately,
spill over onto the dividend growth rate.

We contribute to this literature by presenting a general equilibrium model that an-
alyzes how agents’ characteristics, such as risk aversion and money illusion, affect the
endogenous dividend growth rate by inducing changes in the dividend-price ratio and
expected inflation. Specifically, inspired by the empirical evidence of the aforementioned
papers, we model the dividend growth rate as a linear combination of the endogenous
dividend-price ratio and expected inflation. We derive analytical expressions that link
the model deep parameters to the equilibrium dividend growth rate.

Our main findings can be summarized as follows. First, we show that when the
expected dividend growth rate is determined endogenously, the real short-term interest
rate and the dividend-price ratio can be decreasing in the expected growth rate of the
money supply. These results contrast with the findings of Basak and Yan (2010) that
when agents suffer from money illusion, these equilibrium quantities are increasing in the
expected money supply growth rate. In their model, the real short-term interest rate
comoves with both the expected growth rate of the money supply and dividends (i.e.,
the higher the expected growth rate of the money supply and dividends, the higher the real short-term interest rate). These higher rates prevent agents from borrowing from the future and help them to smooth consumption. In contrast, our equilibrium dividend growth rate is tightly linked to the expected money supply growth rate. Depending on the level of risk aversion and money illusion, the expected growth rate of money supply can negatively impact the expected dividend growth rate. Thus, in an economy where the monetary regime has a higher money supply growth rate, the described mechanism can generate a lower price level, which boosts the future demand for real consumption, causing real bonds to be an attractive investment. As a result, the real short-term interest rate has to decline to induce precautionary savings.

Second, our calibration exercise sheds light on how the exposures of the dividend growth rate to the dividend-price ratio and expected inflation affects its level. In essence, due to the general equilibrium feedback effects, a higher exposure of the dividend growth rate to any of these components ends up affecting the level of the components as well. For instance, when relative risk aversion is relatively small (in between one and two), we show that an increase of the exposure of the equilibrium dividend growth rate to the dividend-price ratio generates simultaneously a small decline in the dividend yield and a sharp decline in expected inflation, resulting in a higher level for the equilibrium dividend growth rate itself. This result finds support in the empirical findings of Chen (2009) and Møller and Sander (2017). Both studies show a significant increase in the exposure of dividend growth to dividend yield from the prewar period to the postwar period, going approximately from $-0.45$ to $-0.01$. These changes were accompanied by an increase in the dividend growth rate from $1.21\%$ to $5.89\%$ and a decline in dividend yield from $1.60\%$ to $1.18\%$.

Third, we show that the dividend-price ratio increases as agents become more illusioned. This result follows from the fact that a more illusioned investor perceives technological shocks as cheaper, while monetary shocks become more expensive. Since the data indicate that the money supply is less volatile than the dividends process, this shift
in the prices of risk makes the real state price density less volatile, which culminates in a less volatile stock price. As a result, the equilibrium stock price declines and boosts the dividend-price ratio. This result is in line with the empirical findings of David and Veronesi (2013) that money illusion tends to generate a higher earnings-price ratio.

On the other hand, the effect of the relative risk aversion parameter on the dividend-price ratio is mixed. We show numerically that higher levels of the risk aversion parameter can potentially decrease the dividend-price ratio if the exposure of the expected dividend growth rate to the dividend-price ratio itself is sufficiently negative. When compared with this finding, we verify that the effects of money illusion and risk aversion on expected inflation appear to be of second-order of importance.

Last, we investigate the general equilibrium implications for the inflation derivatives market. In particular, we complement the existing literature (see, for example, Lioui and Poncet (2005), Sarais (2014), and Dam et al. (2018)) by showing that the price of a zero-coupon inflation caplet increases with risk aversion and decreases with money illusion. The reason is that the risk aversion always increases the total volatility of the real state price density by boosting the market price of technological risk. On the other hand, money illusion introduces the trade-off between technological and monetary shocks; that is, if agents suffer from a higher degree of money illusion, the price of monetary risk becomes more expensive and the price of technological risk becomes cheaper. Thus, if monetary shocks are less volatile than technological shocks, a shift in money illusion that amplifies the importance of the former while reducing the impact of the latter will reduce the total volatility of the real state price density. In summary, an increase in risk aversion increases the total volatility of the discounted payoff of the zero-coupon inflation caplet, making this option more valuable, while money illusion decreases its total volatility, causing the option to be worth less.

The remainder of the article is organized as follows. Section 2 presents the model setup. Section 3 introduces the equilibrium definition and outlines the results. Section 4 presents the calibration of the model with market data and the investigation of the effects
of risk aversion and money illusion on equilibrium quantities. Section 5 concludes. Proofs are presented in the appendix.

2. Model

We consider an infinite horizon continuous-time monetary-based exchange economy, where uncertainty is represented by the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with augmented filtration \(\mathcal{F} = \{\mathcal{F}(t)\}_{t \geq 0}\) generated by two independent Brownian motions, \(Z^\delta = \{Z^\delta(t)\}_{t \geq 0}\) and \(Z^m = \{Z^m(t)\}_{t \geq 0}\), that we refer to as technological and monetary shocks, respectively. All stochastic processes are progressively measurable with respect to \(\mathcal{F}\), and usual integrability conditions are assumed to hold.

Perishable commodity. There is a perishable commodity represented by \(\delta = \{\delta_t\}_{t \geq 0}\). We assume that the commodity expected growth rate is a linear combination of two endogenous quantities: (i) the dividend-price ratio \(q\) and (ii) the expected inflation \(\pi\). Thus, the dynamics of the commodity is represented by the following stochastic differential equation:

\[
\frac{d\delta_t}{\delta_t} = (\epsilon_1 q + \epsilon_2 \pi)dt + \sigma_\delta dZ^\delta_t, \quad \delta_0 > 0,
\]

where \(\epsilon_1 \in \mathbb{R}, \epsilon_2 \in \mathbb{R}\) and \(\sigma_\delta \in \mathbb{R}_+\) are exogenously given. The parameters \(\epsilon_1\) and \(\epsilon_2\) represent the exposure of the dividend growth rate to the factors \(q\) and \(\pi\) to be determined in equilibrium.

We emphasize that our modeling choice differs substantially from the existing exchange economies in the literature. Instead of assuming that the commodity growth rate \(\mu_\delta\) is exogenously set, we postulate that it depends on two equilibrium quantities. Thus, contrary to the standard Lucas (1978) economy, once the equilibrium is characterized, our framework provides an explanation on how the commodity growth rate is affected by changes in the agents’ characteristics, such as risk aversion and money illusion, by observing the responses of the dividend-price ratio \(q\) and expected inflation \(\pi\).
Money supply. The stock of money $M = (M_t)_{t \geq 0}$ is supplied by monetary authorities, and it is assumed to solve the following stochastic differential equation:

$$\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dZ^m_t, \ M_0 \text{ given},$$

where $\mu_m$ and $\sigma_m$ are exogenous positive constants.

The endogenous price of money $p_t$ is postulated to have the following dynamics:

$$\frac{dp_t}{p_t} = \mu_p dt + \sigma_p^\delta dZ^\delta_t + \sigma_p^m dZ^m_t,$$

where $\mu_p, \sigma_p^\delta,$ and $\sigma_p^m$ are determined in equilibrium.\(^1\)

Financial Markets. The risky asset $S = (S_t)_{t \geq 0}$ is in unit supply and represents a claim on the stream of the commodity $\delta$. The real stock price is posited to evolve according to

$$\frac{dS_t + \delta_t dt}{S_t} = \mu_s dt + \sigma_s^\delta dZ^\delta_t + \sigma_s^m dZ^m_t,$$

where the coefficients $\mu_s, \sigma_s^\delta,$ and $\sigma_s^m$ are determined in equilibrium.

Furthermore, there are real and nominal money market funds in zero net supply in this market. Their price processes are represented by $B = (B_t)_{t \geq 0}$ and $B^n = (B^n_t)_{t \geq 0}$, respectively, and we consider them as locally riskless in real and nominal terms. We represent the equilibrium instantaneous real short-term and nominal rate of return by $r$ and $R$. The bonds satisfy the following stochastic differential equations:

$$\frac{dB_t}{B_t} = r dt,$$

$$\frac{dB^n_t}{B^n_t} = (\mu_p + R) dt + \sigma_p^\delta dZ^\delta_t + \sigma_p^m dZ^m_t.$$

\(^1\)We opt to follow Bakshi and Chen (1996) and Basak and Gallmeyer (1999) and model the price of money instead of the consumer price index (CPI). Since the CPI is the inverse of the price of money, once $p_t$ is determined, the CPI is obtained simply by calculating the money price inverse (i.e., $CPI_t = 1/p_t$).
Households. We consider an infinite lived household deriving utility over streams of consumption good $c_t$ and real money holdings $p_tM^d_t$ in the spirit of Bakshi and Chen (1996), Buraschi and Jiltsov (2005), and Lioui and Poncet (2005). We depart from these studies by assuming that the relative risk aversion parameter is larger than one. While their assumption of logarithm agents considerably simplifies the calculations, it also prevents us from understanding the effects of risk aversion on equilibrium quantities.

In addition, we follow Basak and Yan (2010) and assume that households suffer from money illusion (i.e., they are partially confused with real and nominal quantities), which generates a bias towards nominal rates. According to Modigliani and Cohn (1979), this bias emerges from the fact that our daily transactions are conducted in nominal rather than real terms. As a result, agents tend to incorrectly use nominal rates when discounting future real payoffs, generating the mispricing of assets.\(^2\)

The investor’s static variational problem of our model is expressed as

$$\max_{c_t,p_tM^d_t} E \left[ \int_0^\infty e^{-\beta t} \left( \frac{p_t}{p_0} \right)^\ell \left( \frac{c_t^\alpha (p_tM^d_t)^{1-\alpha}}{1-\gamma} \right)^{1-\gamma} dt \right]$$

subject to

$$E \left[ \int_0^\infty \xi_t (c_t + R p_t M^d_t) dt \right] \leq S_0 + E \left[ \int_0^\infty \xi_t R p_t M^d_t dt \right], \quad (1)$$

where $\beta$ is the subjective rate of time preference, $\alpha \in (0,1)$ is the elasticity between consumption and real money demand, and $\ell \in [0,1]$ is the degree of money illusion experienced by investors. When $\ell = 0$, agents do not suffer from money illusion, while $\ell = 1$ represents the case when investors are completely confused between real and nominal rates.

The unique real state price density $\xi_t$ has dynamics given by

$$\frac{d\xi_t}{\xi_t} = -rdt - \theta_d dZ_t^d - \theta_m dZ_t^m, \quad \xi_0 = 1.$$ 

\(^2\)We refer the readers to Basak and Yan (2010) and references therein for an extensive discussion on the microfoundations of money illusion as a belief distortion phenomenon. Other recent papers investigating the effects of money illusion for asset prices are Miao and Xie (2013), David and Veronesi (2013), and Duarte and Saporito (2018).
The market prices of technological and monetary risk, denoted respectively by \( \theta_\delta \) and \( \theta_m \), are determined in equilibrium.\(^3\)

3. Equilibrium

We define the equilibrium in our continuous-time exchange monetary economy as follows.

**Definition 1.** The equilibrium is a set of prices, in real and nominal terms, consisting of the stock price, the money market account price, the dividend growth rate, interest rates, the market prices of risk, and the price of money, represented by \( \{S_t, B_t, B^n_t, \mu_\delta, r, R, \theta_\delta, \theta_m, p_t\} \), and a set of optimal decision rules on consumption good and real money demand, represented by the functions \( \{c_t, p_tM^d_t\} \), such that the representative investor maximizes expected utility in (1), clearing the asset, money, and good markets.

**Proposition 1.** The real state price density is

\[
\xi_t = e^{-\beta t} \left( \frac{\delta_t}{\delta_0} \right)^{\ell - \gamma} \left( \frac{M_0}{M_t} \right)^{\ell}.
\]  

(2)

The real short-term interest rate and market prices of risk are represented by

\[
r = \beta + \ell \mu_m + (\gamma - \ell)\mu_\delta + (1 - \ell + \gamma)(\ell - \gamma)\frac{\sigma^2_\delta}{2} - \ell(1 + \ell)\frac{\sigma^2_m}{2},
\]

(3)

\[
\theta_\delta = (\gamma - \ell)\sigma_\delta,
\]

(4)

\[
\theta_m = \ell \sigma_m,
\]

(5)

where the endogenous expected dividend growth rate is given by

\[
\mu_\delta = \frac{\epsilon_1 \beta + (\epsilon_2 + \ell \epsilon_1)\mu_m + (2\epsilon_2 - \epsilon_1(1 + \ell - \gamma)(\ell - \gamma))\frac{\sigma^2_\delta}{2} - \epsilon_1\ell(1 + \ell)\frac{\sigma^2_m}{2}}{1 + \epsilon_2 + \epsilon_1(1 + \ell - \gamma)}.
\]

(6)

\(^3\)While the standard procedure is to state the dynamic programming problem and show its equivalence to the static variational problem, we opt to directly state the latter to save space. The equivalence between these problems is derived following the exact same steps as in Proposition 2.1 of Basak and Gallmeyer (1999).
Proposition 1 contains several key results. First, contrary to the model of Bakshi and Chen (1996), money is not superneutral (i.e., changes in the money supply affect not only nominal quantities but real variables as well). As revealed by equation (2), the mechanism that prevents the detachment of the financial side of the economy from its real counterpart is money illusion. In the absence of money illusion ($\ell = 0$), the only driver of the real state price density becomes the commodity level, and the unrealistic feature of money neutrality is recovered. Second, the expressions for the market prices of technological and monetary risk in (4) and (5), respectively, show that money illusion introduces a trade-off between these shocks. As investors become more illusioned (i.e., as $\ell$ increases), monetary shocks become more expensive and technological shocks cheaper. Thus, the confused investor associates bad economic times (high marginal utility states) with low money supply and, consequently, shocks that drive the supply up are perceived as bad economic shocks. Simultaneously, the illusioned agent downplays positive shocks to the commodity level. As a result, the higher the degree of illusion $\ell$, the higher the exacerbation of these two effects. A crucial implication of this mechanism is that the total volatility of the real state price density declines with the degree of illusion $\ell$ if the money supply is less volatile than the commodity (i.e., $\sigma_m < \sigma_\delta$). On the other hand, the relative risk aversion $\gamma$ solely affects the price of technological shocks, which implies that a more risk-averse agent always results in a more volatile real pricing kernel.

Third, the expressions for the short-term interest rate in (3) and the dividend growth rate in (6) display high nonlinearity in terms of the relative risk aversion $\gamma$, the degree of money illusion $\ell$, and the exposures $\epsilon_1$ and $\epsilon_2$. Thus, assessing the impact of these parameters on these equilibrium quantities is much more cumbersome than in standard exchange economies. For this reason, we provide a numerical analysis in Section 4 that illustrates the effect of these parameters on equilibrium quantities using market data.

A crucial additional implication that follows from the nonneutrality of money is that the real short-term interest rate depends on the money supply expected growth rate $\mu_m$ and its volatility $\sigma_m$. The next proposition provides an important result implied by this
outcome.

**Proposition 2.** The real short-term interest rate is increasing in the expected growth of the money supply if

\[
\frac{\gamma \epsilon_2 + \ell (1 + \epsilon_1)}{1 + \epsilon_2 + \epsilon_1 (1 + \ell - \gamma)} > 0,
\]

and decreasing otherwise.

Proposition 2 shows that, when the expected dividend growth rate is determined endogenously, the real short-term interest rate is decreasing in the expected growth rate of the money supply, if the inequality in (7) is not satisfied. In contrast with the findings of Basak and Yan (2010) where the short-term rate is increasing in the money supply growth rate, our endogenous determination of the commodity growth rate \(\mu_\delta\) shows that the impact of \(\mu_m\) on \(r\) depends on the signs and magnitude of the exposures \(\epsilon_1\) and \(\epsilon_2\) as well, allowing for a configuration where the expected growth rate of the money supply negatively impacts the expected dividend growth rate. Thus, in an economy where the monetary regime has a higher money supply growth rate, the described mechanism can generate a lower price level, which boosts the future demand for real consumption, causing real bonds to be an attractive investment. As a result, the real short-term interest rate has to decline to induce precautionary savings.

Next, we turn to the characterization of the nominal side of the economy. The next proposition summarizes the results.

**Proposition 3.** The nominal state price density is

\[
\eta_t = e^{-\beta t} \left( \frac{\delta_t}{\delta_0} \right)^{1+\ell-\gamma} \left( \frac{M_0}{M_t} \right)^{1+\ell}.
\]

The nominal short-term interest rate and the nominal market prices of risk are

\[
R = \beta + (1 + \ell) \mu_m + (\gamma - \ell - 1) \mu_\delta - (1 + \ell)(2 + \ell) \frac{\sigma_m^2}{2} - (\ell - \gamma)(1 + \ell - \gamma) \frac{\sigma_\delta^2}{2}.
\]
\[ \theta^n_\delta = (\gamma - \ell - 1)\sigma_\delta, \]
\[ \theta^n_m = (1 + \ell)\sigma_m. \]

The expression for the expected inflation is
\[
\pi = \frac{(1 + \epsilon_1(1 + \ell - \gamma))(\sigma^2_\delta + \mu_m) - \epsilon_1 \left( \beta + \ell \mu_m - (1 + \ell - \gamma)(\ell - \gamma)\sigma^2_\delta - \ell(1 + \ell)\sigma^2_m \right)}{1 + \epsilon_2 + \epsilon_1(1 + \ell - \gamma)}.
\]

The price of money is given by
\[
p_t = \frac{1 - \alpha}{\alpha R} \frac{\delta_t}{M_t}.
\]

The findings of Proposition 3 share some similarities with the results for the real economy. First, the failure of money superneutrality caused by money illusion makes the short-term nominal interest rate affected by the growth rate and the volatility of dividends. This result clearly contrasts with the findings of Bakshi and Chen (1996), where the financial and real economy are completely detached from each other. The trade-off previously discussed between the technological and monetary prices of risk introduced by money illusion is present in the nominal economy as well. The expression for the price of money \( p_t \) is identical to the one in Bakshi and Chen (1996) due to the geometric Brownian motion structure of the commodity level and money supply. Consequently, the price of money in our model shares the exact same characteristics as theirs. Last, the equilibrium expression of expected inflation is a nonlinear function of the relative risk aversion, the degree of money illusion, and the exposures \( \epsilon_1 \) and \( \epsilon_2 \).

To complete the characterization of the equilibrium, we present the expression for the real stock price in our economy in the next proposition.

**Proposition 4.** The real stock price is
\[
S_t = \frac{\delta_t}{q},
\]
where the equilibrium dividend-price ratio $q$ is given by

$$q = \frac{(1 + \epsilon_2) \left( \beta + \ell \mu_m - (1 + \ell - \gamma)(\ell - \gamma)(\sigma_2^2 - \ell(1 + \ell)\frac{\sigma_2^2}{2}) \right) - (1 + \ell - \gamma)(\sigma_2^2 + \mu_m)\epsilon_2}{1 + \epsilon_2 + \epsilon_1(1 + \ell - \gamma)}.$$

Moreover, the dividend-price ratio is increasing in the expected growth of the money supply if

$$\frac{\ell + (\gamma - 1)\epsilon_2}{1 + \epsilon_2 + \epsilon_1(1 + \ell - \gamma)} > 0,$$

and decreasing otherwise.

Proposition 4 contains an insightful result. It shows that the expected money supply growth rate affects the dividend-price ratio $q$, but in contrast with the findings of Basak and Yan (2010), its impact depends on the signs and magnitudes of $\gamma$, $\ell$, $\epsilon_1$, and $\epsilon_2$. When the inequality (8) is not satisfied, the expected growth rate of the money supply $\mu_m$ affects the expected dividend growth rate $\mu_\delta$ positively. As a result, a monetary regime with higher expected money supply growth rate boosts future dividends payments, resulting in a larger present value calculation. Consequently, the stock price is higher and the dividend yield declines.

### 3.1. Inflation Derivatives

Another important contribution of our study is to provide an understanding of how options written on the CPI are affected by changes in risk aversion and money illusion. In particular, we investigate how the price of a zero-coupon inflation caplet changes with the preference parameters.

A zero-coupon inflation caplet consists of a call option written on the inflation rate implied by the CPI during a specific period of time. For simplicity, we assume that the period of reference used to calculate inflation coincides with the life span of the option.
Thus, the payoff of an inflation caplet maturing at \( T \) is simply

\[
\max\left( \frac{CPI_T}{CPI_0} - K, 0 \right).
\]

The put option is referred to as the zero-coupon inflation floorlet, and its payoff is the maximum between zero and the strike minus the implied inflation. The next proposition presents the price of a zero-coupon inflation caplet. We omit the result for the floorlet for the sake of brevity.

**Proposition 5.** The price of a zero-coupon inflation caplet maturing at time \( T \) with strike price \( K \) is

\[
C(0, T, K) = e^{-(2r-R-\sigma_\delta^2-\sigma_m^2)T}\Phi\left(d_1 + \sqrt{((\ell - \gamma - 1)^2\sigma_\delta^2 + (\ell - 1)^2\sigma_m^2)T}\right)
-
Ke^{-rT}\Phi\left(d_1 + \sqrt{(\ell - \gamma)^2\sigma_\delta^2 + \ell^2\sigma_m^2)T}\right),
\]

where

\[
d_1 = \frac{-\ln K + (\mu_m - \mu_\delta + (\sigma_\delta^2 - \sigma_m^2)/2)T}{\sqrt{(\sigma_\delta^2 + \sigma_m^2)T}},
\]

and \( \Phi \) is the standard normal cumulative function distribution.

As shown in Proposition 5, the inflation caplet price displays a highly nonlinear dependency on the degree of money illusion \( \ell \) and the relative risk aversion \( \gamma \). They impact the caplet price in three ways: (i) by changing the real and nominal short-term interest rates \( r \) and \( R \), (ii) by changing the volatility adjustment to \( d_1 \) in the argument of the normal distribution function, and (iii) by changing the equilibrium expected dividends growth rate \( \mu_\delta \) in \( d_1 \). We investigate the price sensitivity to these parameters in the next section.
4. Numerical Analysis

4.1. Data

To investigate the impact of the exogenous parameters $\gamma, \ell, \epsilon_1,$ and $\epsilon_2$ on the equilibrium quantities, we calibrate our model using two datasets. The first dataset comes from Robert Shiller’s website and consists of four time series: the CPI, real earnings, the cyclically adjusted price-earnings ratio (CAPE), and the real rate GS10. The second dataset is the money supply data (M2 series) obtained from the Federal Reserve Bank of St. Louis. We aggregate the original monthly series to a yearly frequency and restrict our analysis to the period 1960–2017, for which data on M2 is available.

We use the earnings series to proxy the dividend process $(\delta_t)_{t \geq 0}$ and the inverse of the CAPE as dividend-price ratio $q$. Following the literature, we interpret the commodity as the output of a mature firm that does not reinvest and pays all earnings as dividends. We calculate the log difference of the dividends, the CPI, and the M2 series. Figure A.1 presents the time series of the log difference in dividends (solid line) and the dividend yield (dashed line) in Panel 1(a) and time series of the log differences of M2 (solid line) and CPI (dashed line) in Panel 1(b).

[Figure 1 about here.]

We obtain the money supply volatility $\sigma_m$ by calculating the standard deviation of the M2 series presented in Panel 1(b). The estimate of the money supply expected growth rate $\mu_m$ is calculated by averaging the log differences of M2 and adjusting it by its quadratic variation term $\sigma_m^2/2$. The same procedure applied to the dividends series and the CPI generates the estimates for the expected dividend growth rate $\mu_\delta$, dividends volatility $\sigma_\delta$, expected inflation $\pi$, and the CPI volatility. The expected growth rate and volatility of these series are summarized in Table A.1.

[Table 1 about here.]
In Section 3, we show that similar to expected inflation $\pi$ and the dividend growth rate $\mu_\delta$, both the real short-term interest rate $r$ and the dividend-price ratio $q$ are constants as well. Thus, we proxy $r$ and $q$ by their sample average and use these quantities in the calibration exercise described next.

4.2. Calibration

The objective of our calibration exercise is to obtain sensible estimates for the parameters $\beta, \gamma, \ell, \epsilon_1, \text{and } \epsilon_2$ and to perform a sensitivity analysis on the neighborhood of these estimates to see how the equilibrium quantities vary accordingly. We set the parameters $\mu_m, \sigma_m, \text{and } \sigma_\delta$ using the estimates presented in Table A.1.

The implied parameters are calibrated by minimizing the distance between the analytical expressions of $r, \mu_\delta, \pi, \text{and } q$, obtained in Propositions 1, 3, and 4, and the historical average of $r$ and $q$ along with the estimates of $\pi$ and $\mu_\delta$ presented in Table A.1. The calibration results are presented in Table A.2.

[Table 2 about here.]

The implied parameters presented in Table A.2 are in line with the literature. The relative risk aversion of 2.29 is in the same range of values considered by Wachter (2013), Campbell et al. (2015), and Albuquerque et al. (2016). The estimate of 0.87 for the degree of money illusion is in line with the estimate of 0.81 from David and Veronesi (2013), while a time preference parameter $\beta$ of 0.053 is in consonance with Campbell and Cochrane (1999).

Checking the results for the exposures $\epsilon_1$ and $\epsilon_2$ is more challenging, since there is no clear benchmark in the literature. The main studies in the field, such as Chen (2009), Engsted and Pedersen (2010), Chen et al. (2012), Möller and Sander (2017), and Zhu et al. (2018), differ on their findings on the sign and magnitude of these coefficients. For this reason, we use the calibration presented in Table A.2 and perform comparative statics in the neighborhood of these values to understand how changes in the model parameters affect the equilibrium quantities.
4.3. Results

We first investigate the sensitivity of the endogenous dividend growth rate $\mu_\delta$ to its exposures to the dividend-price ratio $\epsilon_1$ and to expected inflation $\epsilon_2$. As shown in Propositions 3 and 4, in equilibrium, the expressions for the dividend-price ratio $q$ and expected inflation $\pi$ depend on $\epsilon_1$ and $\epsilon_2$ as well. Thus, the expected dividend growth rate is highly nonlinear on the exposures $\epsilon_1$ and $\epsilon_2$. Figure A.2 shows the comparative statics of $\mu_\delta$ relative to these parameters.

[Figure 2 about here.]

Figure A.2 illustrates that for any positive level of $\epsilon_2$, the expected dividend growth rate $\mu_\delta$ is an increasing function of its exposure to dividend-price ratio $\epsilon_1$. The reason is that an increase in $\epsilon_1$ is accompanied by a boost in the dividend-price ratio $q$ and a decline in expected inflation. The economics behind this mechanism is as follows. When the exposure of the expected dividend growth rate to dividend yield $\epsilon_1$ increases, the expected higher future dividend payments are associated with lower future marginal utility. Hence, the future dividend payments due to a stock investment have lower values. As a result, the stock return needs to increase and stock price to drop to maintain market clearing. Ultimately, the lower stock price level boosts the dividend-price ratio $q$. At the same time, since the CPI and the dividend level have an inverse relationship, the higher the future expected dividend growth, the lower the future expected CPI growth rate (i.e., the lower expected inflation $\pi$). Thus, the effect of an increase in $\epsilon_1$ on each component of the expected dividend growth rate $\mu_\delta$ is summarized as

$$\mu_\delta = \epsilon_1 q + \epsilon_2 \pi.$$

A similar analysis can help us to understand the sensitivity of $\mu_\delta$ relative to $\epsilon_2$. Contrary to the previous case, $\mu_\delta$ is an increasing function of $\epsilon_2$ for low values of $\epsilon_1$ and a decreasing function of $\epsilon_2$ for high values of $\epsilon_1$ in the investigated range. To understand the mechanism driving these results, consider first the case where the expected growth rate
has a large negative exposure to the dividend-price ratio. In this region, an increase in $\epsilon_2$ is accompanied by an increase in the dividend-price ratio $q$ and a decline in expected inflation $\pi$ by the same economic rationale described as before. However, different from the previous analysis, the exposure $\epsilon_1$ is fixed at a low (negative) value. Hence, the increase in $q$ generated by a higher exposure to expected inflation $\epsilon_2$ acts as a decreasing force in the equilibrium expected dividend growth rate. Nevertheless, the impact on the expected inflation component of the equilibrium expected dividend growth rate prevails, which generates an upward sloping curve for $\mu_\delta$. The impact is now summarized as

$$\mu_\delta = \epsilon_1 q + \epsilon_2 \pi.$$  

In the other case, where $\epsilon_1$ is fixed at a large (positive) value, expected inflation becomes negative (i.e., deflation). Thus, an increase in $\epsilon_2$ reduces the expected dividend growth. Lower future prospects for dividends are associated with future states of high marginal utility. Hence, future dividends payments made by the stock have high values, which boosts the price of the stock and, consequently, damps the dividend yield $q$. Once again, the inverse relationship between the CPI and the dividend level implies that the lower future expected dividend growth is associated with a higher future expected CPI growth rate (i.e., a higher expected inflation $\pi$). Thus, these effects are summarized as

$$\mu_\delta = \epsilon_1 q + \epsilon_2 \pi.$$  

The overall effect of an increase $\epsilon_2$ on the expected dividend growth rate $\mu_\delta$ is negative, since the last term is negative because $\pi < 0$ in this region, as shown in Figure A.2.

Figure A.3 shows the impact of relative risk aversion and money illusion on the dividend-price ratio and expected inflation for three different levels of the exposure $\epsilon_1$. The solid line corresponds to the benchmark $\epsilon_1$ from Table A.2. The dashed line represents the case $\epsilon_1 = -0.101$, while the dash-dotted line depicts $\epsilon_1 = 0.099$. As illustrated
in the right plot of Panel 3(a), the dividend-price ratio $q$ increases with the degree of money illusion $\ell$. The result follows from the trade-off between technological and monetary shocks induced by money illusion, as discussed in Proposition 1. As investors become more illusioned, monetary shocks become pricier and technological shocks cheaper, making discounted payoffs less volatile and the stock cheaper. Consequently, dividend yield increases, as shown in the graph.

The previous result differs from the effect of risk aversion on $q$, as illustrated by the plot on the left of Panel 3(a). For low levels of relative risk aversion ($\gamma$ approximately between one and two), the dividend yield is decreasing in $\epsilon_1$. In fact, Chen (2009) and Møller and Sander (2017) find that the significant increase in $\epsilon_1$ from the prewar period to the postwar period from $-0.45$ to $-0.01$ was accompanied by an increase in the dividend growth rate from 1.21% to 5.89% and a decline in dividend yield $q$ from 1.60% to 1.18%.

For the benchmark case, the dividend yield is increasing in $\gamma$. As agents become more risk averse, the expected dividend growth rate increases and the stock price has to adjust downward for markets to clear, culminating in a higher dividend-price ratio. The crucial mechanism here is that, despite the fact that a more risk-averse agent always increases the total volatility of discounted payoffs, it also affects the expected growth rate of dividends. Consequently, depending on the magnitude of the exposure $\epsilon_1$, the increase in the expected dividend growth rate generated by a larger relative risk aversion parameter counteracts the effects of higher volatility levels, causing the stock price to depreciate and lower the dividend yield. Panel 3(b) complements the analysis by showing that the effects of the preference parameters on expected inflation are mild compared to the dividend-price ratio effects.

[Figure 3 about here.]

Last, Figure A.4 presents the sensitivity of a zero-coupon inflation caplet price to changes in risk aversion (left plot) and money illusion (right plot) for three different maturities: five years (dashed line), 15 years (solid line), and 30 years (dash-dotted line).
Figure A.4 shows that relative risk aversion and money illusion affect caplet prices in different ways. As discussed in Proposition 1, relative risk aversion only affects the exposure of the real pricing kernel to technological shocks. Hence, the more risk averse the agent becomes, the higher the market price of technological risk, which results in higher total volatility of the discounted payoffs, leading the call option to be worth more. On the other hand, money illusion introduces an exposure trade-off between technological and monetary shocks. As the investor becomes more confused between nominal and real quantities, the real pricing kernel becomes more exposed to monetary shocks and less exposed to technological shocks. As a result, if the money supply is less volatile than the commodity, as the data suggest in Table A.1, the shift from dividends to the money supply in the real pricing kernel makes discounted payoffs less volatile. Thus, the total volatility of the discounted payoffs reduces and the call option is worth less.

5. Conclusion

Our paper studies how money illusion and risk aversion affect real and financial quantities when the expected growth rate of dividends is determined endogenously. Under this framework, we show that money illusion breaks the superneutrality of money (i.e., money supply affects real equilibrium quantities, a drawback commonly present in continuous-time monetary-based economies). In addition, contrary to the existing studies that assume an exogenous specification of the dividend growth rate, we show that the monotonicity of the real short-term interest rate \( r \) and the dividend-price ratio \( q \) relative to the money supply growth rate \( \mu_m \) depends on the exposures of the dividend growth rate to expected inflation \( \epsilon_1 \) and to the dividend-price ratio \( \epsilon_2 \), as well as the money illusion parameter \( \ell \) and the relative risk aversion \( \gamma \).

We also provide an analysis of how the zero-coupon caplets are affected by the agent’s characteristics. We show that the caplet price is increasing in the risk aversion parameter.
due to an increase in the total volatility of the discounted payoff, while the increase in money illusion reduces the caplet price when the money supply volatility is smaller than dividends volatility.

Our study provides several venues for theoretical and empirical explorations. First, a more complex stochastic process can be used to better describe the dynamics of the money supply and the volatility of dividends. While the analytical tractability of the model is certain to be lost, simulations can provide additional insights into a richer equilibrium environment. For instance, a stochastic growth rate of the money supply can generate an implied volatility smile and help researchers understand the impact of money illusion on this curve. Moreover, the persistence of this stochastic component introduces predictability of option returns that can be tested empirically.
Appendix A.

Proof of Propositions 1, 3, and 4. The representative agent solves

$$\max_{c,p,M^d} \mathbb{E} \left[ \int_0^\infty e^{-\beta t} \left( \frac{p_t}{p_0} \right)^\ell \left( \frac{c_t}{c_0} \right)^{\alpha(1-\gamma)} \frac{(p_t M_t^d)^{(1-\alpha)(1-\gamma)}}{1-\gamma} dt \right]$$

s.t. \( \mathbb{E} \left[ \int_0^\infty \xi_t(c_t + R_t p_t M_t^d) dt \right] \leq S_0 + \mathbb{E} \left[ \int_0^\infty \xi_t R_t p_t M_t dt \right]. \)

The first order conditions imply that

$$\alpha e^{-\beta t} \left( \frac{p_t}{p_0} \right)^\ell c_t^{\alpha(1-\gamma)-1} \left( p_t M_t^d \right)^{(1-\alpha)(1-\gamma)} = y \xi_t, \quad (A.1)$$

$$(1-\alpha) e^{-\beta t} \left( \frac{p_t}{p_0} \right)^\ell c_t^{\alpha(1-\gamma)} \left( p_t M_t^d \right)^{(1-\alpha)(1-\gamma)-1} = y \xi_t R_t. \quad (A.2)$$

Dividing (A.2) by (A.1), we obtain the following expression for the nominal short-term interest rate

$$R_t = \frac{1 - \alpha}{\alpha} \frac{c_t}{p_t M_t^d}. \quad (A.2)$$

Similarly, dividing (A.1) by the expression itself evaluated at time zero, leads to

$$\xi_t = e^{-\beta t} \left( \frac{p_t}{p_0} \right)^\ell \frac{c_t^{\alpha(1-\gamma)-1} \left( p_t M_t^d \right)^{(1-\alpha)(1-\gamma)}}{c_0^{\alpha(1-\gamma)} \left( p_0 M_0^d \right)^{(1-\alpha)(1-\gamma)}}. \quad (A.3)$$

Using the good and money market clearing conditions, we have

$$\xi_t = e^{-\beta t} \left( \frac{p_t}{p_0} \right)^{\ell+(1-\alpha)(1-\gamma)} \left( \frac{\delta_t}{\delta_0} \right)^{\alpha(1-\gamma)-1} \left( \frac{M_t}{M_0} \right)^{(1-\alpha)(1-\gamma)} \xi_t. \quad (A.3)$$

Define \( \xi_{t,v} \equiv \xi_t \). The no-arbitrage condition on the endogenous price of money implies that

$$p_t = \mathbb{E}_t \left[ \int_t^\infty \xi_{t,v} p_v R_v dv \right].$$
Using the conjecture \( p_v = A\delta_v M_v \), where \( A, a, b \) are constants to be determined, we have that

\[
p_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\beta(v-t)} \left( \frac{p_v}{p_t} \right) ^\ell \left( \frac{\delta_v}{\delta_t} \right) ^{\alpha(1-\gamma)-1} \left( \frac{p_v M_v}{p_t M_t} \right) ^{(1-\alpha)(1-\gamma)} \frac{1 - \alpha \delta_v}{\alpha M_v} \, dv \right] \\
= \frac{1 - \alpha \delta_t}{\alpha M_t} \mathbb{E}_t \left[ \int_t^\infty e^{-\beta(v-t)} \left( \frac{p_v}{p_t} \right) ^{\ell+(1-\alpha)(1-\gamma)} \left( \frac{\delta_v}{\delta_t} \right) ^{\alpha(1-\gamma)} \left( \frac{M_v}{M_t} \right) ^{(1-\alpha)(1-\gamma)-1} \, dv \right] \\
= \frac{1 - \alpha \delta_t}{\alpha M_t} \left[ \int_t^\infty e^{-\beta(v-t)} \left( \frac{\delta_v}{\delta_t} \right) ^{(1-\gamma)+a(\ell+(1-\alpha)(1-\gamma))} \left( \frac{M_v}{M_t} \right) ^{(1-\alpha)(1-\gamma)-1+b(\ell+(1-\alpha)(1-\gamma))} \, dv \right]
\]

where

\[
\lambda = \beta - (\alpha(1-\gamma) + a(\ell + (1 - \alpha)(1 - \gamma))) (\mu_\delta - \sigma_\delta^2/2) - (\alpha(1-\gamma) + a(\ell + (1 - \alpha)(1 - \gamma)))^2 \sigma_\delta^2/2 \\
- ((1 - \alpha)(1 - \gamma) - 1 + b(\ell + (1 - \alpha)(1 - \gamma))) (\mu_m - \sigma_m^2/2) \\
- ((1 - \alpha)(1 - \gamma) - 1 + b(\ell + (1 - \alpha)(1 - \gamma)))^2 \sigma_m^2/2.
\]

Thus, \( A = (1 - \alpha)/(\alpha \lambda) \), \( a = 1 \) and \( b = -1 \), and the price of money can be written as

\[
p_t = \frac{1 - \alpha \delta_t}{\alpha \lambda M_t} \tag{A.4}
\]

Substituting the values of \( a \) and \( b \) into \( \lambda \), we obtain

\[
\lambda = \beta - (1 + \ell - \gamma) \left( \mu_\delta + (\ell - \gamma) \frac{\sigma_\delta^2}{2} \right) + (1 + \ell) \left( \mu_m - \left( 1 + \frac{\ell}{2} \right) \sigma_m^2 \right).
\]

Substituting (A.4) into (A.3), the expression for real state price density becomes

\[
\xi_t = e^{-\beta t} \left( \frac{\delta_t}{\delta_0} \right) ^{\ell-\gamma} \left( \frac{M_0}{M_t} \right) ^\ell \tag{A.5}
\]

An application of Ito’s formula on (A.4) gives the following dynamics for the price of
Since the CPI is the inverse of the price of money $p_t$, an application of Ito’s lemma on $CPI_t = \frac{\alpha \lambda}{1 - \alpha} M_t$ gives

$$\frac{dCPI_t}{CPI_t} = \left( \mu_m + \sigma^2_\delta - \epsilon_1 q - \epsilon_2 \pi \right) dt + \sigma_m dZ^m_t - \sigma_\delta dZ^\delta_t. \quad (A.6)$$

The drift of (A.6) represents the expected inflation. Thus, solving the fixed-point problem for $\pi$, we obtain

$$\pi = \mu_m + \sigma^2_\delta - \epsilon_1 q - \epsilon_2 \pi,$$

$$\pi = \sigma^2_\delta - \epsilon_1 q + \mu_m \frac{1 + \epsilon_2}{1 + \epsilon_2}. \quad (A.7)$$

Note that once the dividend-price ratio $q$ is determined in terms of the exogenous parameters of the model, the whole equilibrium is fully characterized. Thus, to characterize this quantity, we turn to the expression of the real stock price. Under no-arbitrage, we write

$$S_t = \mathbb{E}_t \left[ \int_t^\infty \xi_{t,v} \delta_v dv \right] = \mathbb{E}_t \left[ \int_t^\infty e^{-\beta(v-t)} \left( \frac{\delta_v}{\delta_t} \right)^{\ell-\gamma} \left( \frac{M_v}{M_t} \right)^{-\ell} \delta_v dv \right]$$

$$= \delta_t \mathbb{E}_t \left[ \int_t^\infty e^{-\beta(v-t)} \left( \frac{M_v}{M_t} \right)^{-\ell} dv \right] = \frac{\delta_t}{\varrho},$$

where

$$\varrho = \beta + \ell \mu_m - (1 + \ell - \gamma)(\epsilon_1 q + \epsilon_2 \pi) + (\gamma - \ell)(1 + \ell - \gamma) \frac{\sigma^2_\delta}{2} - \ell(1 + \ell) \frac{\sigma^2_m}{2}.$$
Thus, the endogenous dividend-price ratio is \( q = \delta_t/S_t = \varrho \). Solving the equation for \( q \), we have

\[
q = \beta + \ell \mu_m - (1 + \ell - \gamma)(\epsilon_1 q + \epsilon_2 \pi) + (\gamma - \ell)(1 + \ell - \gamma) \frac{\sigma_d^2}{2} - \ell(1 + \ell) \frac{\sigma_m^2}{2}.
\]

(A.8)

Substituting (A.7) into (A.8) and solving for \( q \) gives the following expression for the dividend-price ratio

\[
q = \frac{\beta + \ell \mu_m - (1 + \ell - \gamma)\epsilon_2 \pi + (\gamma - \ell)(1 + \ell - \gamma) \frac{\sigma_d^2}{2} - \ell(1 + \ell) \frac{\sigma_m^2}{2}}{1 + \epsilon_1(1 + \ell - \gamma)}. \tag{A.9}
\]

The endogenous expected inflation expression is obtained by substituting (A.9) into (A.7), which leads to

\[
\pi = \frac{(1 + \epsilon_1(1 + \ell - \gamma))\sigma_d^2 + \mu_m - \epsilon_1 \left( \beta + \ell \mu_m - (1 + \ell - \gamma)(\ell - \gamma) \frac{\sigma_d^2}{2} - \ell(1 + \ell) \frac{\sigma_m^2}{2} \right)}{1 + \epsilon_2 + \epsilon_1(1 + \ell - \gamma)}. \tag{A.10}
\]

This conclude the characterization of \( q \) and \( \pi \) in terms of the exogenous parameters of the model. Consequently, the expression for the endogenous dividend growth rate is fully determined as well and given by

\[
\mu_d = \epsilon_1 q + \epsilon_2 \pi = \frac{\epsilon_2(\sigma_d^2 + \mu_m) + \epsilon_1 \left( \beta + \ell \mu_m - (1 + \ell - \gamma)(\ell - \gamma) \frac{\sigma_d^2}{2} - \ell(1 + \ell) \frac{\sigma_m^2}{2} \right)}{1 + \epsilon_2 + \epsilon_1(1 + \ell - \gamma)}.
\]

The complete characterization of the dividend process dynamics allow us to determined the real short-term interest rate and the real market prices of risk. A straightforward
application of Ito’s lemma on (A.5) gives

\[ r = \beta + \ell \mu_m + (\gamma - \ell)\mu_\delta + (1 - \ell + \gamma)(\ell - \gamma)\frac{\sigma_\delta^2}{2} - \ell(1 + \ell)\frac{\sigma_m^2}{2}, \]

\[ \theta_\delta = (\gamma - \ell)\sigma_\delta, \]

\[ \theta_m = \ell \sigma_m. \]

To conclude, we use the characterization of the state price density in (A.5) and the price of money in (A.4) to calculate the price of the zero-coupon bond in real and nominal terms. It follows that the real bond price maturing at time \( T \) is

\[ B(t, T) = \mathbb{E}_t [\xi_{t,T}] = \mathbb{E}_t \left[ e^{-\beta(T-t)} \left( \frac{\delta_T}{\delta_t} \right)^{\ell-\gamma} \left( \frac{M_T}{M_t} \right)^{-\ell} \right] = e^{-r(T-t)}. \]

The nominal bond price maturing at time \( T \) is

\[ B^n(t, T) = \mathbb{E}_t \left[ \xi_{t,T} \frac{p_T}{p_t} \right] = \mathbb{E}_t \left[ e^{-\beta(T-t)} \left( \frac{\delta_T}{\delta_t} \right)^{1+\ell-\gamma} \left( \frac{M_t}{M_T} \right)^{1+\ell} \right] = e^{-R(T-t)}, \]

where the nominal short-term interest rate is

\[ R = \beta + (1 + \ell)\mu_m + (\gamma - \ell - 1)\mu_\delta - (1 + \ell)(2 + \ell)\frac{\sigma_m^2}{2} - (\ell - \gamma)(1 + \ell - \gamma)\frac{\sigma_\delta^2}{2}. \]

\[ \Box \]

**Proof of Proposition 2.** Substituting (6) into (3) and grouping the terms multiplying \( \mu_m \) gives the coefficient

\[ \frac{\gamma \epsilon_2 + \ell (1 + \epsilon_1)}{1 + \epsilon_2 + \epsilon_1 (1 + \ell - \gamma)}, \]

which concludes the proof. \[ \Box \]
strike price $K$ is

$$C(0, T, K) = \mathbb{E} \left[ \xi_{0, T} \left( \frac{\text{CPI}_T}{\text{CPI}_0} - K \right) ^+ \right]$$

$$= \mathbb{E} \left[ e^{-\beta T} \left( \frac{\delta_T}{\delta_0} \right)^{-\gamma} \left( \frac{M_T}{M_0} \right)^{-\ell} \left( \frac{\text{CPI}_T}{\text{CPI}_0} - K \right) ^+ \right]$$

$$= \mathbb{E} \left[ e^{-\beta T} \left( \frac{\delta_T}{\delta_0} \right)^{-\gamma} \left( \frac{M_T}{M_0} \right)^{-\ell} \frac{\text{CPI}_T}{\text{CPI}_0} \mathbb{I} \{ \frac{\text{CPI}_T}{\text{CPI}_0} > K \} \right] - KE \left[ e^{-\beta T} \left( \frac{\delta_T}{\delta_0} \right)^{-\gamma} \left( \frac{M_T}{M_0} \right)^{-\ell} \mathbb{I} \{ \frac{\text{CPI}_T}{\text{CPI}_0} > K \} \right].$$

Using the sets equivalence

$$\{ \frac{\text{CPI}_T}{\text{CPI}_0} > K \} \iff \left\{ \dfrac{\delta_0 M_T}{\delta_T M_0} > K \right\}$$

$$\iff \left\{ (\mu_m - \mu_d + \frac{\sigma_m^2 - \sigma_d^2}{2})T + \sigma_m Z_T^m - \sigma_d Z_T^d > \ln K \right\}$$

$$\iff \left\{ z \sqrt{(\sigma_m^2 + \sigma_d^2)} T > \ln K + \left( \mu_d - \mu_m + \frac{\sigma_m^2 - \sigma_d^2}{2} \right)T \right\}$$

$$\iff \{ z > -d_1 \},$$

where $z \sim N(0, 1)$ and

$$d_1 = -\ln K + (\mu_m - \mu_d + \frac{\sigma_m^2 - \sigma_d^2}{2})T \sqrt{(\sigma_m^2 + \sigma_d^2)T},$$

the first part becomes

$$\mathbb{E} \left[ e^{-\beta T} \left( \frac{\delta_T}{\delta_0} \right)^{-\gamma} \left( \frac{M_T}{M_0} \right)^{-\ell} \frac{\text{CPI}_T}{\text{CPI}_0} \mathbb{I} \{ \frac{\text{CPI}_T}{\text{CPI}_0} > K \} \right] = \mathbb{E} \left[ e^{-\beta T} \left( \frac{\delta_T}{\delta_0} \right)^{-\gamma - 1} \left( \frac{M_T}{M_0} \right)^{-\ell} \mathbb{I} \{ \frac{\text{CPI}_T}{\text{CPI}_0} > K \} \right]$$

$$= e^{-\beta (1-\ell)\mu_m - (\ell-\gamma-1)\mu_d + (\ell-\gamma-1)\sigma_m^2/2 + (1-\ell)\sigma_m^2/2} T \mathbb{E}_4 \left[ e^{(\ell-\gamma-1)\sigma_d Z_T^d + (1-\ell)\sigma_d Z_T^d} \mathbb{I} \{ \frac{\text{CPI}_T}{\text{CPI}_0} > K \} \right]$$

$$= e^{-\beta (1-\ell)\mu_m - (\ell-\gamma-1)\mu_d + (\ell-\gamma-1)\sigma_m^2/2 + (1-\ell)\sigma_m^2/2} T \int_{-d_1}^\infty e^{((\ell-\gamma-1)\sigma_d Z_T^d + (1-\ell)\sigma_d Z_T^d)z - z^2/2} \sqrt{2\pi} dz$$

$$= e^{-\beta (1-\ell)\mu_m - (\ell-\gamma-1)\mu_d - (\ell-\gamma-1)(\ell-\gamma-2)\sigma_d^2/2 + (1-\ell)(\ell-\gamma-2)\sigma_d^2/2} T d_1 + \sqrt{((\ell-\gamma-1)^2\sigma_d^2 + (1-\ell)^2\sigma_m^2)T}$$

$$= e^{-2\gamma - \sigma_d^2 - \sigma_m^2} T \Phi \left( d_1 + \sqrt{((\ell-\gamma-1)^2\sigma_d^2 + (1-\ell)^2\sigma_m^2)T} \right).$$
The second part can be written as

\[- KE_t \left[ e^{-\beta T} \left( \frac{\delta_T}{\delta_0} \right)^{\ell-\gamma} \left( \frac{M_T}{M_0} \right)^{-\ell} \mathbb{I} \{ \frac{\text{CPIT}}{\text{CPI}} > K \} \right] \]

\[= -Ke^{-(\beta+\ell \mu_\delta+\ell)\mu_\delta+\ell(\gamma-\ell)\sigma^2_\delta/2-\ell\sigma^2_m/2}T E_t \left[ e^{(\ell-\gamma)\sigma Z^2_\delta-\ell\sigma_m Z^2_m} \mathbb{I} \{ \frac{\text{CPIT}}{\text{CPI}} > K \} \right] \]

\[= -Ke^{-(\beta+\ell \mu_\delta+\ell)\mu_\delta+\ell(\gamma-\ell)\sigma^2_\delta/2-\ell\sigma^2_m/2} \int_{-d_1}^\infty e^{\sqrt{(T-t)\sigma^2_\delta+\ell\sigma^2_m} (T-t)z-z^2/2} \Phi \left( d_1 + \sqrt{(\ell-\gamma)^2 \sigma^2_\delta + \ell^2 \sigma^2_m} T \right) dz \]

\[= -Ke^{-\gamma T} \Phi \left( d_1 + \sqrt{(\ell-\gamma)^2 \sigma^2_\delta + \ell^2 \sigma^2_m} T \right), \]

which concludes the proof. \(\square\)


Gabriele Sarais. Pricing inflation and interest rates derivatives with macroeconomic foundations. 2014.


Figure A.1: Data. The figures show the time series of log differences in dividends ($\Delta \log \delta$, solid line) and the dividend yield ($q$, dashed line) in the top and the time series of log differences in the money supply ($\Delta \log M_2$, solid line) and the log differences in the CPI ($\Delta \log CPI$, dashed line) for the period 1960–2017. The scales are shown in percentage terms and the data is reported in annual frequency.
Figure A.2: Expected dividend growth rate. The graph shows the effect of the exposures $\epsilon_1$ and $\epsilon_2$ on the expected dividend growth rate. Using the empirical findings of Chen (2009) and Engsted and Pedersen (2010) as references, we vary $\epsilon_1$ in the interval $[-1, 1]$ and $\epsilon_2$ in $[0, 1]$. The results of $\mu_5$ are shown in percentage terms.
(a) Effect of risk aversion and money illusion on dividend-price ratio.

(b) Effect of risk aversion and money illusion on expected inflation.

Figure A.3: Equilibrium dividend yield and expected inflation. The panels show the effect of risk aversion and money illusion in the equilibrium dividend yield (in percentage) and expected inflation (in basis points). The effects are analyzed for three different levels of the exposure $\epsilon_1$. 

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Figure A.4: Zero-coupon inflation caplet. The graphs illustrate the effect of relative risk aversion (left plot) and money illusion (right plot) on the price of a zero-coupon caplet price for three different maturities: five years (dashed line), 15 years (solid line), and 30 years (dash-dotted line). The strike price \( K \) is fixed at 1.01. The parameters used in the price evaluation are presented in Tables A.1 and A.2, while we vary \( \gamma \) in the interval [1, 5] and \( \ell \) in [0, 1].
Parameter values in percentage terms.

<table>
<thead>
<tr>
<th>Series</th>
<th>Expected Growth Rate</th>
<th>Volatility</th>
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</thead>
<tbody>
<tr>
<td>Dividends</td>
<td>0.25 ($\mu_d$)</td>
<td>2.56 ($\sigma_d$)</td>
</tr>
<tr>
<td>M2</td>
<td>0.54 ($\mu_m$)</td>
<td>0.23 ($\sigma_m$)</td>
</tr>
<tr>
<td>CPI</td>
<td>0.3 ($\pi$)</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Table A.1: Descriptive Statistics. The table shows the expected growth rate and volatility of dividends, money supply, and the CPI. We assume all three series follow a geometric Brownian motion with constant coefficients, as described in the text. The expected growth rate is calculated by averaging the log differences of the time series presented in Figure A.1 and adjusting by their quadratic variation. The volatilities are calculated as the standard deviation of the series.
**Implied parameters in annual terms.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Relative risk aversion $\gamma$</td>
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<tr>
<td>Money illusion $\ell$</td>
<td>0.87</td>
</tr>
<tr>
<td>Rate of time preference $\beta$</td>
<td>0.053</td>
</tr>
<tr>
<td>Exposure to dividend-price ratio $\epsilon_1$</td>
<td>$-0.001$</td>
</tr>
<tr>
<td>Exposure to expected inflation $\epsilon_2$</td>
<td>0.891</td>
</tr>
</tbody>
</table>

**Equilibrium quantities in percentage terms.**

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>5.89</td>
<td>5.87</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0.33</td>
<td>0.31</td>
</tr>
<tr>
<td>$\mu_5$</td>
<td>0.29</td>
<td>0.24</td>
</tr>
<tr>
<td>$r$</td>
<td>6.03</td>
<td>6.11</td>
</tr>
</tbody>
</table>

Table A.2: Calibration results. The top part of the table presents the estimates for the implied parameters in annual terms. The bottom part compares the model fit with the data counterpart.